So now you've got Signed Graphs

How Zaslavsky can work for you

Oscar Coppola, Mikey Reilly

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The Main Result

Recall the result of our last talk:

Theorem (Stanley's Acyclicity Theorem)

For any graph G with n vertices, the number of acyclic orientations of G is $(-1)^n\chi_G(-1)$, where χ_G is the chromatic polynomial of G.

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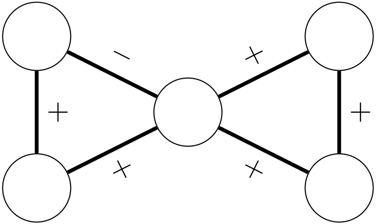
Theorem (Zaslavsky's Acyclicity Theorem)

For any signed graph Σ with n vertices, the number of acyclic orientations of Σ is $(-1)^n\chi_\Sigma(-1)$, where χ_Σ is the chromatic polynomial of Σ.

What is a Signed Graph?



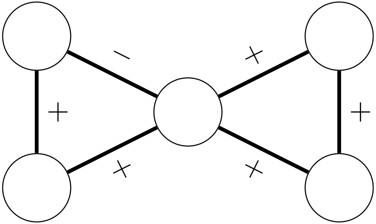
Signed graphs assign a "positive" or "negative" sign to each edge.

This is a simple case of a more general construction, where edges of a graph are labeled with elements of your favorite group. Such graphs are called *voltage graphs*. We will consider only signed graphs, which are voltage graphs under the group $\mathbb{Z}_2 = \{+, -\}$.

Colorings

This begs the question, what do the signed edges actually mean?

We will see that signed edges allow for more complex and richer interactions between colorings and orientations.



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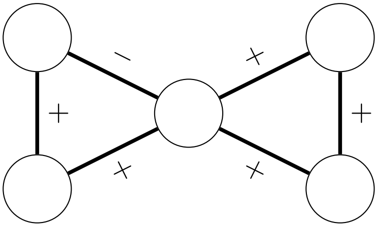

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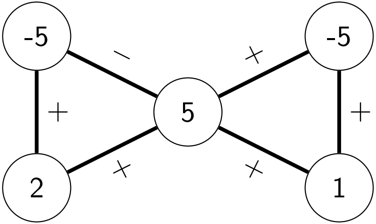

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Proper Colorings

Formally, a signed λ-coloring on Σ is a function κ : vertices of $\Sigma \to \{-\lambda, \ldots, \lambda\}$ for some $\lambda \in \mathbb{N}$.

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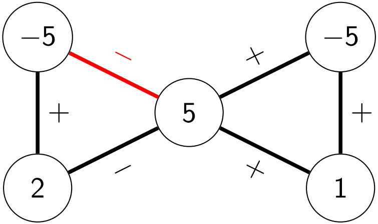
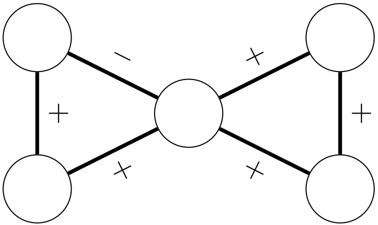
This means that in a proper coloring, endpoints of an edge cannot be the same color if the edge is positive and they cannot be negatives of each other is the edge is negative.



Chromatic Polynomials



If we were to ignore all of the signs and treat Σ as an unsigned graph, then its chromatic polynomial is $(\lambda)(\lambda-1)^2(\lambda-2)^2$, i.e. there are this may ways to properly color the graph in the colors $\{1, 2, \ldots, \lambda\}$.

What if we repeat the same process with signed colorings instead?

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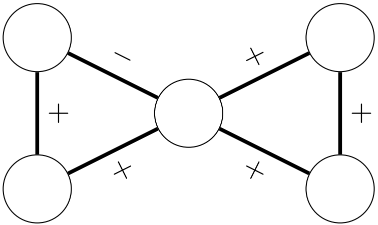

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What if we repeat the same process with signed colorings instead?

There are $(2\lambda)^4(2\lambda-1)$ ways to color Σ in the colors $\{-\lambda, \ldots, \lambda\}$.

Chromatic Polynomials

For the purpose of proving our main result, we will focus solely on colorings which could possibly include zero and so we define the *signed chromatic polynomial*: $\chi_{\Sigma}(\lambda)$ to be the number of proper colorings of Σ.

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Deletion for signed and unsigned graphs is the same. However, there is a catch when it comes contractions of signed graphs.

Contractions

If the edge is positive, it contracts and the two vertices are merged as usual.

If the edge is negative, we'll need to *switch* one of the vertices to make the edge positive in order to merge.

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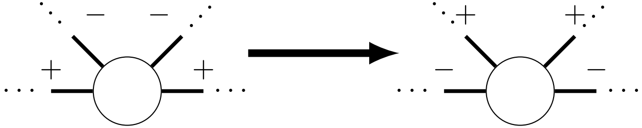
If the edge is negative, we'll need to *switch* one of the vertices to make the edge positive in order to merge.

Switching a vertex means flipping the sign of each edge touching the vertex.



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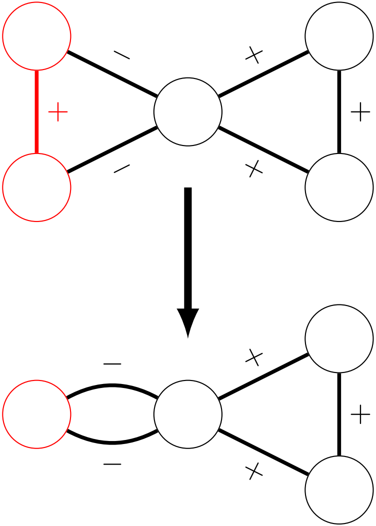

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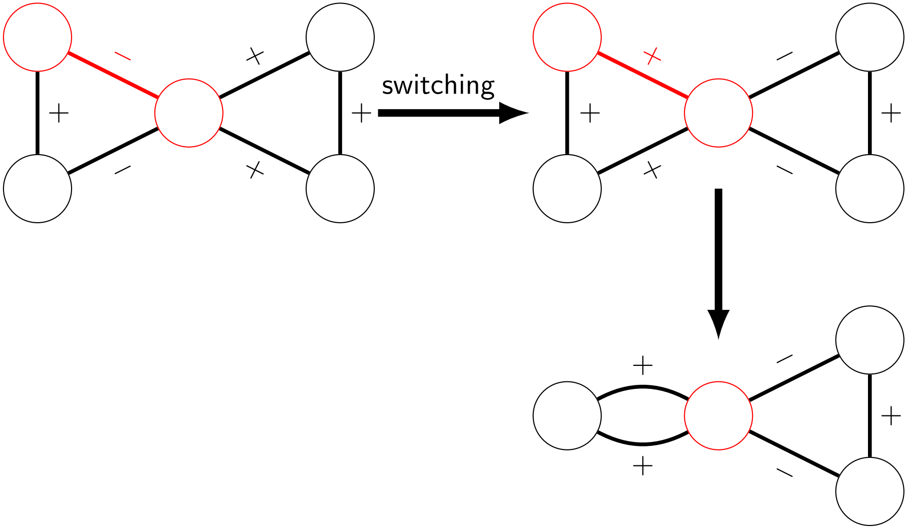
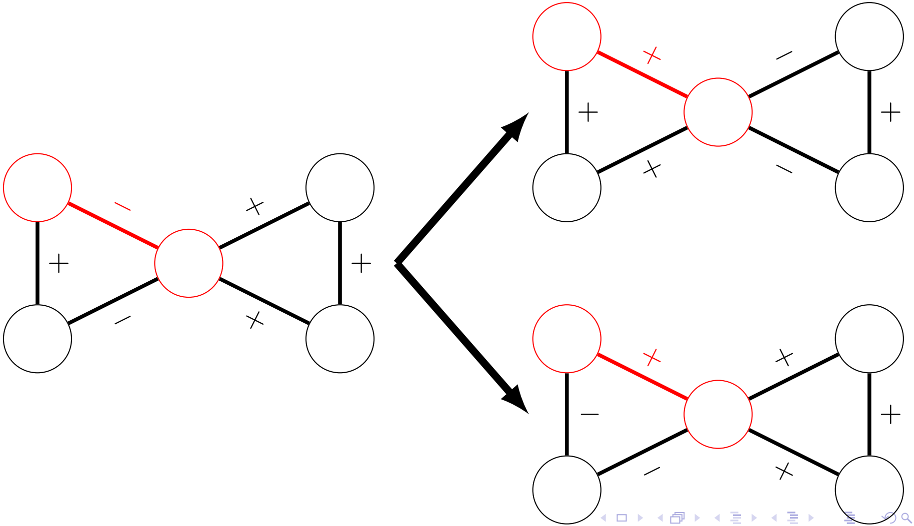


Contractions

But wait! Switching introduces a choice: we could switch either of the two vertices before contracting. Is this okay?



Contractions

While we do get superficially different graphs, the choice of which vertex to switch does not change the number of valid colorings, which is what we care about.

In fact, we obtain a simple bijection when we use the convention that switching a vertex also flips its color (from positive to negative or vice versa).

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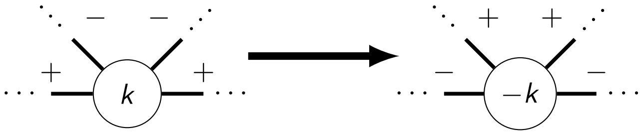

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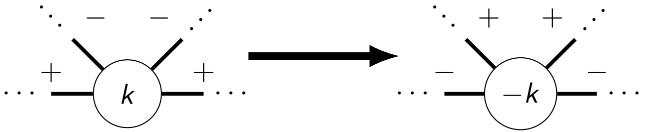

A coloring is proper after switching iff it is proper before switching. For our talk, we'll consider a contraction to be valid only if the endpoints are the same color (up to sign) before.

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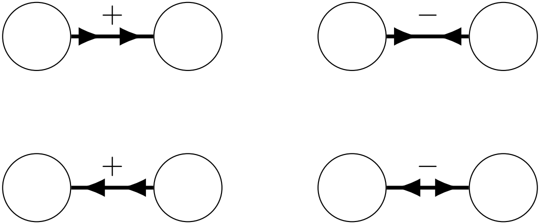
Signed orientations can come in two flavors, depending on the sign of the edge:



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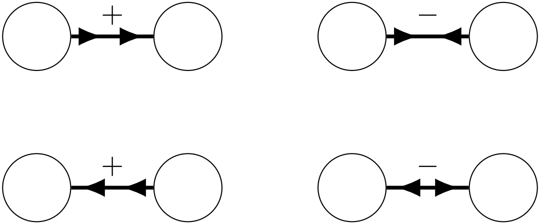

These have direct analogues to normal directed edges, through a secret weapon.

The Graph between Worlds

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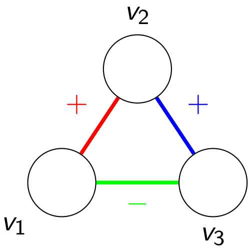

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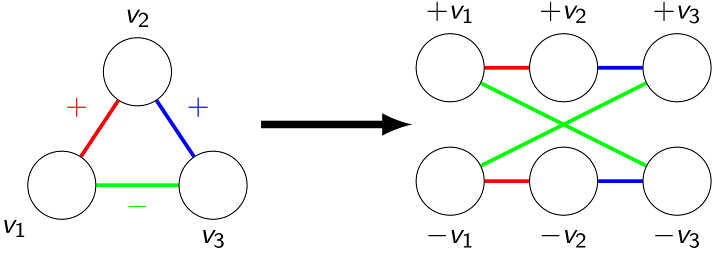

The covering graph of Σ, call it $\overline{\Sigma}$, has two vertices and two edges for each vertex and edge of Σ.

Covering the Rules

How can we explicitly construct the covering graph?

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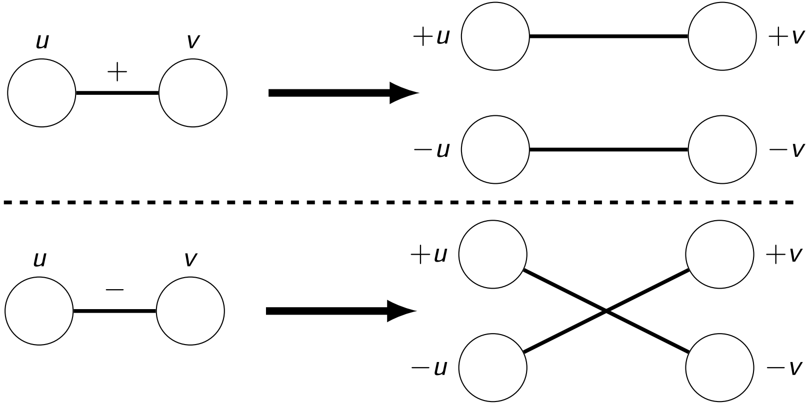
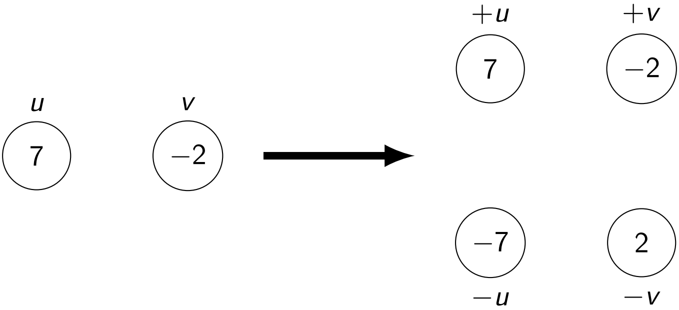
For each vertex v in Σ, we create vertices $+v$ and $-v$ in $\overline{\Sigma}$. Now suppose that there is an edge between u and v in Σ. Then we want an edge between $+u$ and $\text{sgn}(e)v$ and an edge between $-u$ and $-\text{sgn}(e)v$ in $\overline{\Sigma}$.



Coloring the Rules

Suppose that we have a graph Σ along with a coloring κ. We can use κ to create a coloring on $\overline{\Sigma}$ by coloring $+v$ by $\kappa(v)$ and $-v$ by $-\kappa(v)$.



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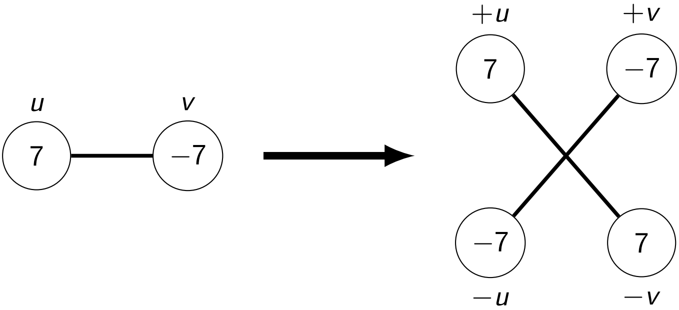

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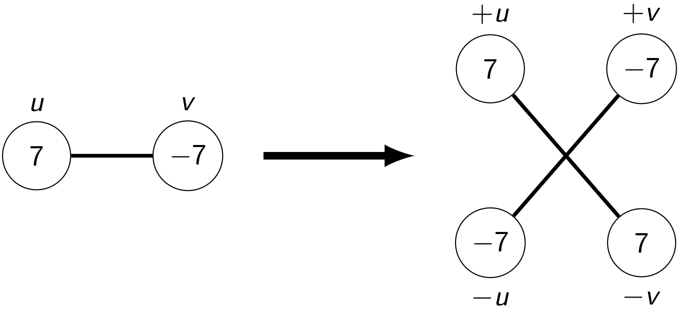

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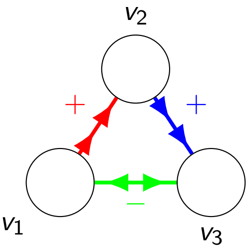
And what about orientations?

Signed covering graphs turn signed directed edges into normal directed edges.



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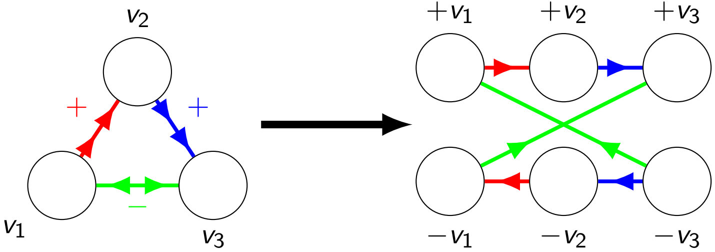
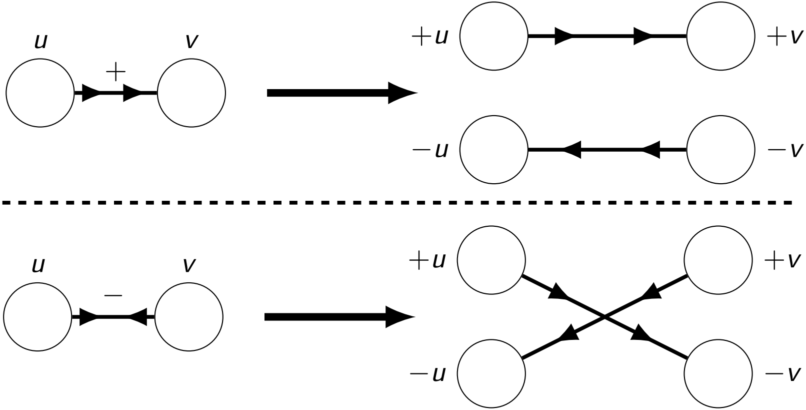


And what about orientations?

In reality, orientations are assigned to each half-edge on the covering graph. Half-edges attached to $+u$ match the signed graph's orientation, and are reversed for $-u$. In this way, half-edges will always point in the same direction on the covering graph.



And what about orientations?

So, we can reliably encode the information in directed signed graphs as directed unsigned graphs. This lets us motivate the following definition.

Definition

A circuit C in Σ is cyclic iff every vertex in C is neither a "source" nor a "sink" relative to edges in C.

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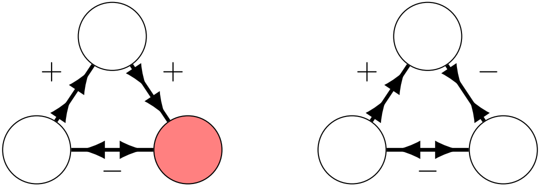

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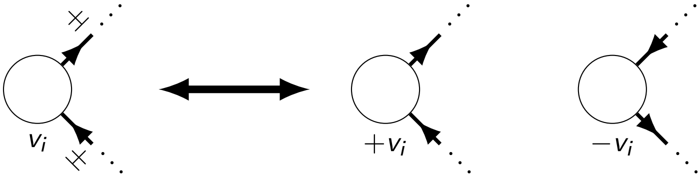
A signed graph Σ is cyclic iff its signed covering graph $\overline{\Sigma}$ is cyclic.



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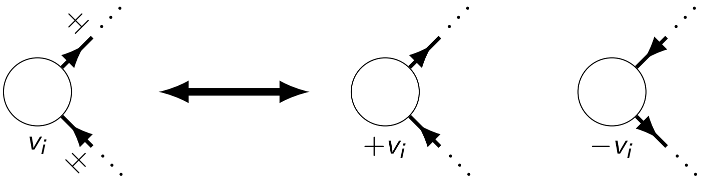

The orientation alone determines the covering's orientation, and vice versa. So, no sources or sinks are introduced either way.

Zaslavsky's Acyclicity Theorem

Theorem (Zaslavsky's Acyclicity Theorem)

For any signed graph Σ with n vertices, the number of acyclic orientations of Σ is $(-1)^n \chi_\Sigma(-1)$, where χ_Σ is the chromatic polynomial of Σ.

The proof strategy will be almost identical to the proof of Stanley's Acyclicity Theorem.

Compatible Pairs

A signed λ-coloring κ along with an acyclic orientation τ are a *compatible pair* for Σ if τ is acyclic and if the induced coloring and orientation on the covering graph $\overline{\Sigma}$ has the property that all arrows point from larger colors to smaller (or equal) colors. The covering graph view shows that under this coloring condition, any cycle must be colored improperly.

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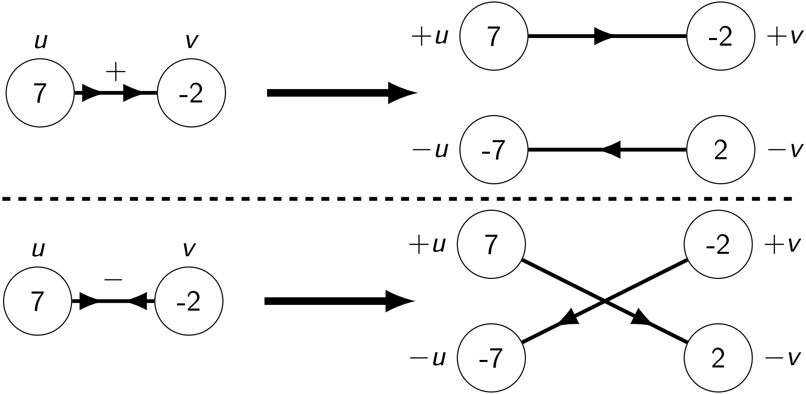

Nothing much, what's υp with you?

Define $\upsilon_\Sigma(2\lambda + 1)$ (which is only defined for odd arguments) to be the number of compatible pairs in the colors $\{-\lambda, \ldots, \lambda\}$.

Note that $\upsilon_\Sigma(1) = \upsilon_\Sigma(2 \cdot 0 + 1)$ is the number of acyclic orientations of Σ since there is only one way to color the entire graph by the single color 0.

In order to prove that $(-1)^n \chi_\Sigma(-1)$ is really the number of acyclic orientations of Σ, we will show that $\chi_\Sigma(-\lambda) = (-1)^n \upsilon_\Sigma(\lambda)$, where n is the number of vertices in Σ.

Deja Vu

It turns out that the signed chromatic polynomial has very similar defining properties to the regular chromatic polynomial, namely:

(1) $\chi_{\circ}(\lambda) = \lambda$ for odd λ

(2) $\chi_{\Sigma_1 \sqcup \Sigma_2}(\lambda) = \chi_{\Sigma_1}(\lambda) \cdot \chi_{\Sigma_2}(\lambda)$

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We will show that υ has the following properties:

(1') $\upsilon_{\circ}(\lambda) = \lambda$
(2') $\upsilon_{\Sigma_1 \sqcup \Sigma_2}(\lambda) = \upsilon_{\Sigma_1}(\lambda) \cdot \upsilon_{\Sigma_2}(\lambda)$
(3') $\upsilon_{\Sigma}(\lambda) = \upsilon_{\Sigma - e}(\lambda) + \upsilon_{\Sigma / e}(\lambda)$

Once we have shown that these properties hold, then it must be that $(-1)^n \upsilon_{\Sigma}(\lambda) = \chi_{\Sigma}(-\lambda)$ for all odd λ.

Clearly,

The proofs for the first two conditions are elementary and identical to the unsigned case.

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So all that is left is to show that $\upsilon_G(\lambda) = \upsilon_{G-e}(\lambda) + \upsilon_{G/e}(\lambda)$, and once we have shown this, we will be done.

Reverse, Reverse

Pick any (not necessarily compatible) pair (κ, τ) for Σ and pick any edge e in Σ. Let τ^e be the same as the orientation τ, except the direction on the edge e is reversed.

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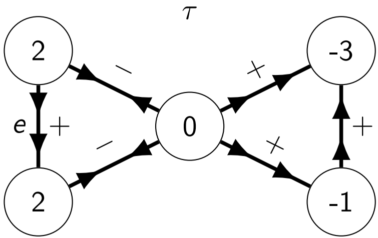
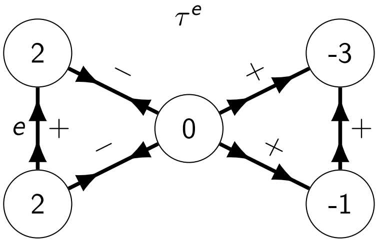


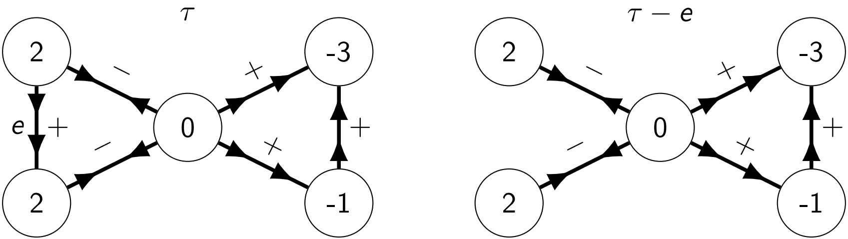


Take it back now

Lemma

If exactly one/both/neither of the pairs (κ, τ), (κ, τ^e) are compatible, then exactly one/both/neither of the pairs $(\kappa, \tau - e)$, $(\kappa/e, \tau/e)$ are compatible, respectively.

Here, $\tau - e$ means what you would expect, it is an orientation on $\Sigma - e$ that agrees with τ on all of the remaining edges.



Notation

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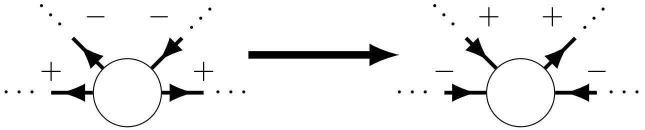
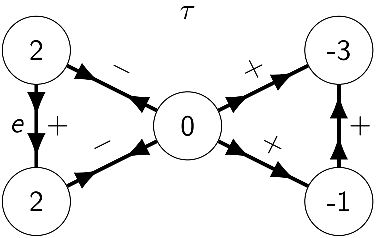
If it exists, $(\kappa/e, \tau/e)$ is a compatible pair for Σ/e. The pair exists only when the endpoints of e are improperly colored, in which case κ/e of the new contracted vertex has the color of each of the endpoints of e, and τ/e agrees with τ on all of the remaining edges.

(Note that during contraction, we possibly need to switch a vertex. When we switch, we need to flip all of arrows adjacent to the vertex.)



Yay More Pictures

An example of a contracted compatible pair.

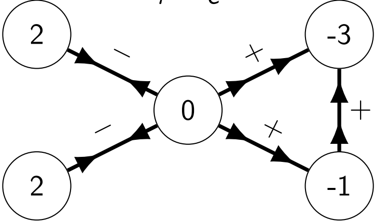


Both are good

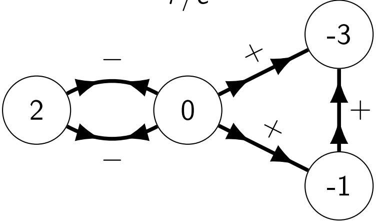
First, let's suppose that both (κ, τ) and (κ, τ^e) are compatible. Then the endpoints of e must be improperly colored, meaning that $(\kappa/e, \tau/e)$ exists.

It must be that both $(\kappa, \tau - e)$ and $(\kappa/e, \tau/e)$ are compatible since neither deletion nor contraction will create cycles.

$\tau - e$



τ/e



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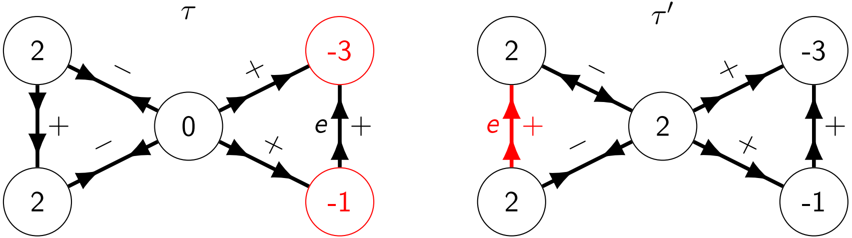
(κ, τ^e) could be incompatible for two reasons. Either the endpoints of e are properly colored or (κ, τ^e) contains a cycle.



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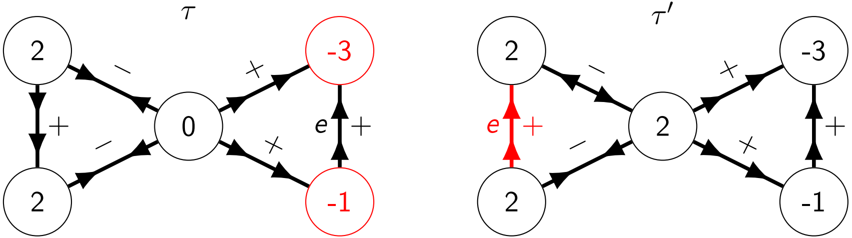

In either case, $(\kappa/e, \tau/e)$ will not be a compatible pair. However, $(\kappa, \tau - e)$ will be a compatible pair with no issue.

Contradiction

Finally, suppose that neither of the pairs (κ, τ), (κ, τ^e) are compatible, and assume (for the sake of contradiction) that it is due to e; that is, the presence of e violates either the coloring or acyclic condition of compatibility.

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Well, the endpoints of e must be improperly colored, since otherwise the direction which points toward the lower color would be compatible with κ. So, it must be that both τ and τ^e are cyclic, and that they contain different cycles (since both pass through e).

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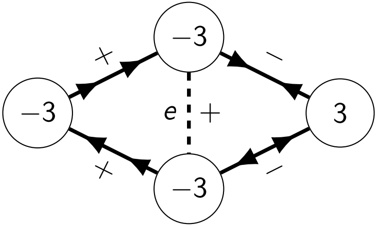

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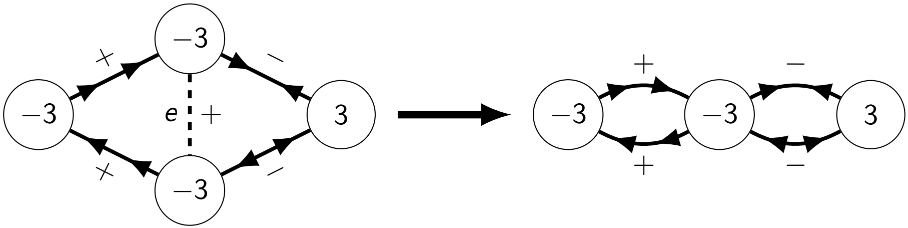
The contraction τ/e would contain both corresponding cycles, meaning that τ/e contains more cycles than τ, which is a contradiction since contraction cannot add more cycles to a graph.



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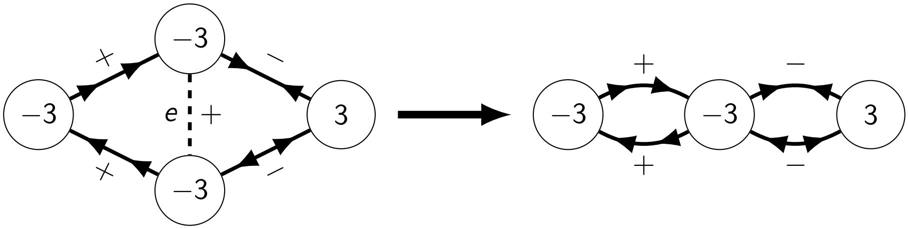

Therefore both (κ, τ), (κ, τ^e) must be incompatible due to edges other than e, and so both $(\kappa, \tau - e)$ and (if it exists) $(\kappa/e, \tau/e)$ will be incompatible.

Not our combinatorics

Therefore, if exactly one/both/neither of the pairs (κ, τ), (κ, τ^e) are compatible, then exactly one/both/neither of the pairs $(\kappa, \tau - e)$, $(\kappa/e, \tau/e)$ are compatible, respectively.

This means that there is a bijection between the compatible pairs for Σ and the compatible pairs for $\Sigma - e$ along with the compatible pairs for Σ/e.

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Recall that this gives us the fact that $(-1)^n \upsilon_\Sigma(\lambda) = \chi_\Sigma(-\lambda)$ for odd λ, and so $(-1)^n \chi_\Sigma(-1) = \upsilon_\Sigma(1) =$ the number of acyclic orientations of Σ.

Source (or Sink)

T. Zaslavsky, Signed graph coloring, Discrete Mathematics 39(2) (1982) 215–228.